

Some identification problems for integro-differential operator equations *

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Abstract. We consider, in a Hilbert space H , the convolution integro-differential equation $u''(t) - h * Au(t) = f(t)$, $0 \leq t \leq T$, $h * v(t) = \int_0^t h(t-s)v(s) ds$, where A is a linear closed densely defined (possibly selfadjoint and/or positive definite) operator in H . Under suitable assumptions on the data we solve the inverse problem consisting of finding the kernel h from the extra data (measured data) of the type $g(t) := (u(t), \varphi)$, where φ is some eigenvector of A^* . An inverse problem for the first-order equation $u'(t) - l * Au(t) = f(t)$, $0 \leq t \leq T$, is also studied when A enjoys the same properties as in the previous case.

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1 Introduction

Let $A : \mathcal{D}(A) \rightarrow H$ be a closed linear operator densely defined in a Hilbert space H with scalar product (\cdot, \cdot) , norm $\|\cdot\|$, and let $\mathcal{D}(A)$ be the domain of definition of A . Let us assume that there exists $\varphi \in \mathcal{D}(A^*)$ such that

$$A^*\varphi = \lambda_0\varphi, \tag{1.1}$$

$$\lambda_0 \in \mathbf{R} \setminus \{0\}, \tag{1.2}$$

$$\|\varphi\| = 1. \tag{1.3}$$

Remark 1.1. One may assume that $\lambda_0 \in \mathbf{C} \setminus \{0\}$: our method remains valid in this case as well. If λ_0 is complex, then one has to replace it by $\overline{\lambda_0}$ in several formulas, such as (2.1), (2.2), (2.4), (2.5).

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Assumptions (1.1)–(1.3) are satisfied if A is selfadjoint, $A = A^*$, and A has an eigenvalue $\lambda_0 \neq 0$. For example, this property holds if A is an elliptic semibounded from below selfadjoint operator in a bounded (smooth) domain. In this case the spectrum of A is discrete and consists of a sequence of real eigenvalues $\{\lambda_n(A)\}_{n=1}^{+\infty}$ going to $+\infty$.

Consider the direct problem

$$u''(t) = \int_0^t h(t-s)Au(s)ds + f(t), \quad 0 \leq t \leq T, \quad (1.4)$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad (1.5)$$

where $T > 0$, and

$$h \in C([0, T]) = C([0, T]; \mathbf{R}), \quad f \in C^1([0, T]; H), \quad (1.6)$$

$$u_0 \in \mathcal{D}(A), \quad u_1 \in H, \quad (u_0, \varphi) \neq 0. \quad (1.7)$$

We emphasize that in equation (1.6) operator A appears only under the integral sign. In other words, we are concerned with the case when the differential operator outside the integral does *not* dominate the operator inside the integral.

Assume now that the kernel h is a real-valued function. This assumption is used only in section 4, formula (4.13). If it is discarded, the part of section 4, which is based on this assumption, should be changed. For example, if one assumes that h is sufficiently small, then the denominator of (4.12) does not vanish for any $j \in \mathbf{N}$ and lemma 4.1 remains valid without the assumption about real-valuedness of h .

Suppose now that for some $h \in C([0, T])$ problem (1.4)–(1.5) has a unique solution

$$u \in C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A)). \quad (1.8)$$

The assumption $h \in C([0, T])$ is used in section 2 (see equation (2.3)) to identify h . Uniqueness of the solution to the identification problem (IP), formulated below formula (1.9), is proved for $h \in C([0, T])$ in section 3.

Assume that the kernel h is unknown and the data

$$g(t) = (u(t), \varphi), \quad 0 \leq t \leq T, \quad (1.9)$$

are measured.

The IP (identification problem) we study is: *given the data $(A, \varphi, u_0, u_1, f, g)$, with $g \in C^2([0, T])$, find a pair (u, h) satisfying (1.4), (1.5), (1.9).*

We note also that the exact data satisfy the additional conditions

$$g(0) = (u_0, \varphi), \quad g'(0) = (u_1, \varphi), \quad g''(0) = (f(0), \varphi). \quad (1.10)$$

If the the function $g(t)$ is not considered as an exact datum, that is, a function of the form (1.9) where u solves (1.4)–(1.5) then conditions (1.10) are necessary for a solution to problem (1.4), (1.5), (1.9) to exist.

We notice that the inverse problem we are going to study differs from the ones studied in [9] and, e.g., [1], [2], and the very approach to our identification problem is new.

Consider now the more general equation depending on the *negative* parameter μ :

$$u''(t) = \mu Au(t) + \int_0^t h(t-s)Au(s) ds + f(t), \quad 0 \leq t \leq T. \quad (1.11)$$

Observe that our identification problem can be viewed as the *limit* case of the identification problem (1.11), (1.5), (1.9) as $\mu \rightarrow 0-$. We recall that such problems have been widely studied under the assumption that A is selfadjoint and positive definite, cf., e.g., [1]–[8].

The main result, proved in section 2, is:

Theorem 1.1. *If (1.1) and (1.3) hold, then the IP has at most one solution.*

An algorithm for finding h and u from the data is described in section 2.

We can deal also with the first-order identification problem consisting in determining a pair of functions $u : [0, T] \rightarrow H$ and $l : [0, T] \rightarrow \mathbf{R}$ satisfying the Cauchy problem

$$u'(t) = l * Au(t) + f(t), \quad 0 \leq t \leq T, \quad (1.12)$$

$$u(0) = u_0. \quad (1.13)$$

as well as the additional conditions

$$g(t) = (u(t), \varphi), \quad 0 \leq t \leq T, \quad l(0) \leq 0, \quad (1.14)$$

where

$$f \in C^2([0, T]; H), \quad u_0 \in \mathcal{D}(A), \quad g(0) = (u_0, \varphi) \neq 0, \quad g \in C^3([0, T]). \quad (1.15)$$

Moreover, we assume that our data satisfy the additional conditions

$$g(0) = (u_0, \varphi), \quad g'(0) = (f(0), \varphi). \quad (1.16)$$

Such conditions are necessary for a solution to problem (1.12)–(1.14) to exist. For exact data they are satisfied automatically.

Differentiating both sides in (1.12) and taking (1.13) into account, it is immediate to deduce that problem (1.12)–(1.14) is equivalent to problem (1.11), (1.5), (1.9) with $\mu = l(0)$, $h = l'$, f being replaced with f' and $u_1 = f(0)$. Since the case corresponding to $l(0) < 0$ has already been studied in the literature, as was mentioned above, we can restrict ourselves to the study of the case $l(0) = 0$. So conditions (1.14) can be replaced with the more specific one

$$g(t) = (u(t), \varphi), \quad 0 \leq t \leq T, \quad l(0) = 0. \quad (1.17)$$

Finally we observe that a problem of the same type as (1.4), (1.5), (1.9) can be treated similarly for the more general equation:

$$u''(t) = A_0 u(t) + \int_0^t h(t-s)Au(s) ds + f(t), \quad 0 \leq t \leq T. \quad (1.18)$$

Here $A_0 : \mathcal{D}(A_0) \rightarrow H$ is a linear closed operator such that

$$\mathcal{D}(A_0) \supset \mathcal{D}(A), \quad \varphi \in \mathcal{D}(A_0^*). \quad (1.19)$$

Further assume that the additional information

$$g_0(t) = (u(t), A_0^* \varphi), \quad 0 \leq t \leq T, \quad (1.20)$$

is available. We can consider the identification problem IP_0 related to equations (1.18), (1.5), (1.9), (1.20) and to the data $(A, A_0, \varphi, u_0, u_1, f, g, g_0)$. Under the assumptions similar to those of theorem 1.1, one can uniquely and algorithmically recover functions h and u from the data. However, the existence of u cannot be guaranteed, since the identification problem (1.18), (1.5), (1.9), (1.20) is, in general, *overdetermined*. Yet, the existence can be proved if, e.g., φ is a common eigenvector to A and A_0 .

We now describe the plan of the paper:

Section 2 is devoted to the existence and uniqueness of the unknown kernel h .

In section 3 the existence and uniqueness of the solution to the direct problem (1.4), (1.5), with general closed selfadjoint operators satisfying (1.1)–(1.3), are proved under suitable assumptions on the data.

In section 4 a mixed initial and boundary value problem is posed for the operator equation (1.4) under the assumption that A is a (closed) selfadjoint and positive definite operator. Such a problem is solved under the assumptions that the Fourier coefficients of the data decay sufficiently fast. The results so found are then applied to the first-order equation $u' - l * Au = f$ with $l(0) = 0$.

In section 5 some applications to linear integro-partial equations are considered.

2 Uniqueness of the solution to IP

In this section we prove Theorem 1.1.

Proof. Multiply both sides of (1.4) by φ and use properties (1.1) and (1.3) to get:

$$g''(t) - \lambda_0 \int_0^t h(t-s)g(s) ds = \psi(t), \quad \psi(t) = (f(t), \varphi), \quad 0 \leq t \leq T. \quad (2.1)$$

Equation (2.1) can be written as a linear Volterra integral equation for h :

$$\int_0^t h(t-s)g(s) ds = \frac{g''(t) - \psi(t)}{\lambda_0} := p(t), \quad 0 \leq t \leq T. \quad (2.2)$$

Differentiate (2.2) and use (1.6) and (1.7) to get

$$g(0)h(t) + \int_0^t g'(t-s)h(s) ds = p'(t), \quad 0 \leq t \leq T. \quad (2.3)$$

Note that the left-hand side of (2.2) is differentiable even when $h \in C([0, T])$, because $g \in C^2([0, T])$. Also, equation (2.2) shows that $p(t) \in C^1([0, T])$, even when

$g \in C^2([0, T]; H)$, because the left-hand side of (2.2) and $\psi(t)$ are differentiable. Therefore, assuming that $g \in C^2([0, T]; H)$, and $f \in C^1([0, T]; H)$, implies, via equation (2.2), that $g \in C^3([0, T]; H)$.

Since $g(0) \neq 0$ by assumptions (1.7) and (1.10), equation (2.3) is a Volterra second-kind equation with continuous kernel g' . Therefore it is uniquely solvable in $C([0, T])$, since, according to our assumptions on the data, $p' \in C([0, T])$.

It is well-known, that the solution to (2.3) can be obtained by iterations, or, in analytic form, by the Laplace transform if one assumes $T = +\infty$. It is also well-known that for any Volterra operator V we have $(cI + V)^{-1} = c^{-1}I + V_1$ if $c = \text{const} \neq 0$, V_1 being, in turn, a Volterra operator.

We have assumed that the solution to (1.4), (1.5), with a known kernel $h \in C([0, T])$, does exist and is unique. Therefore, if h is found (from (2.3)), then u is uniquely found from (1.4), (1.5). Thus, theorem 1.1 is proved. \square

An algorithm for the recovery of h and u from the data consists of solving (2.3) for h and then, once h is known, solving (1.4), (1.5) for u .

Existence and uniqueness of the solution to (1.4), (1.5) is studied in section 3.

Finally, let us discuss problem IP_0 related to equations (1.9), (1.5), (1.18). Multiply (1.18) by φ and get

$$g''(t) - g_0(t) - \lambda_0 \int_0^t h(t-s)g(s) ds = \psi(t), \quad 0 \leq t \leq T, \quad (2.4)$$

where ψ is defined in (2.1). This equation can be reduced to equation (2.3) with function q' replacing p' , where

$$q(t) = \frac{g''(t) - g_0(t) - \psi(t)}{\lambda_0}, \quad 0 \leq t \leq T. \quad (2.5)$$

Thus h is uniquely determined from the data for IP_0 . If one assumes that φ is also an eigenvector of A_0^* with an eigenvalue $\lambda_{0,0}$, then the function g_0 in formulas (2.4), (2.5) can be replaced by $\lambda_{0,0}g$.

A different approach to a study of (2.1) is given in section 4: rewrite first (2.1) as

$$[g(t) - g(0) - tg'(0) - \psi^{(1)}(t)]/\lambda_0 = g * h^{(1)}(t), \quad 0 \leq t \leq T, \quad (2.6)$$

where the superscript (1) stands for convolution of a function r with t , i.e.

$$r^{(1)}(t) = \int_0^t (t-s)r(s) ds, \quad 0 \leq t \leq T. \quad (2.7)$$

Then solve the Volterra equation of the first kind (2.6) for $h^{(1)}$, as it has been done above. If $h^{(1)}$ is found, then $h = (h^{(1)})''$.

To conclude this section we deal with problem (1.12)–(1.14). As before one proves that $l \in C^1([0, T])$ is uniquely defined from the data. Indeed, from (1.12) and (1.14) one derives that

$$g'(t) = \lambda_0 l * g(t) + \psi(t), \quad \psi(t) = (f(t), \varphi), \quad 0 \leq t \leq T, \quad (2.8)$$

where λ_0 is defined in (1.1)-(1.2). The case of complex $\lambda_0 \in \mathbf{C} \setminus \{0\}$ can be treated similarly, as explained in section 1.

From (2.8) one gets:

$$g * l(t) = [g'(t) - \psi(t)]/\lambda_0 := w(t), \quad 0 \leq t \leq T. \quad (2.9)$$

Moreover, since (2.9) is a Volterra equation of the first kind for the unknown $l \in C^1([0, T])$, and it has at most one solution. Since $g(0) \neq 0$, and $w \in C^1([0, T])$, then differentiating (2.9) one gets the second kind Volterra equation for l .

$$g(0)l(t) + g' * l(t) = w'(t), \quad 0 \leq t \leq T. \quad (2.10)$$

This yields the existence and uniqueness of l and an algorithm for the recovery of l , since the second kind Volterra equation can be solved by iterations.

Remark 2.1. From formulas (2.10), (2.9) and (2.8) we easily compute the initial value of l :

$$l(0) = \frac{g''(0) - (f'(0), \varphi)}{g(0)\lambda_0}. \quad (2.11)$$

Hence, necessary conditions for equation (2.9) to admit a solution satisfying $l(0) = 0$ are:

$$g'(0) = (f(0), \varphi), \quad g''(0) - (f'(0), \varphi) = 0. \quad (2.12)$$

3 Existence and uniqueness of the solution to the direct problem (1.4), (1.5)

Let us assume that $h \in C([0, T])$ is *known*, A is selfadjoint and E_λ is its resolution of the identity. The subspace $H_\Lambda = E_\Lambda$ is invariant with respect to A and $\|A\|_{H_\Lambda} \leq \Lambda$.

Assume that the following hypothesis holds:

$$H1 \quad u_0, u_1 \in H_\Lambda, \quad f \in C([0, T]; H_\Lambda).$$

Applying E_Λ to (1.4), using $H1$ and denoting $u_\Lambda(t) = E_\Lambda u(t)$, $t \in [0, T]$, one gets equations (1.4) and (1.5) for u_Λ . Problem (1.4), (1.5) in H_Λ with a bounded operator A , $\|A\|_{H_\Lambda} \leq \Lambda$ in H_Λ , is easily seen to be uniquely solvable, so existence and uniqueness of u_Λ follow. If the hypothesis $H1$ holds with some Λ , then it holds for H_μ with any $\mu > \Lambda$, because $H_\Lambda \subset H_\mu$ for $\mu > \Lambda$. Therefore existence and uniqueness of the solution u to (1.4), (1.5) is proved in any H_μ , $\mu > \Lambda$, provided that the hypothesis $H1$ holds.

This implies uniqueness of the solution to (1.4), (1.5) in H if $H1$ holds. Indeed, assuming there are two solutions u_1 and u_2 to problem (1.4), (1.5), one concludes from the above argument that $\|u_1 - u_2\|_{H_\mu} = 0$ for all $\mu \geq \Lambda$. Since $0 = \lim_{\mu \rightarrow +\infty} \|u_1 - u_2\|_{H_\mu} = \|u_1 - u_2\|_H$, it follows that $u_1 = u_2$.

Our argument proves that *the homogeneous direct problem (1.4), (1.5) has only the trivial solution.*

Since $f = 0$, $u_0 = 0$ and $u_1 = 0$ satisfy $H1$, problem (1.4), (1.5) with a selfadjoint A and $h \in C^2([0, T])$ has at most one solution in H . Indeed, if it has two solutions, their difference, u , solves the homogeneous problem (1.4), (1.5). Consequently $u(t) = \int_0^t h^{(1)}(t-s)Au(s)ds$, where $h^{(1)}(t)$ is defined by formula (2.7) with r replaced by h .

Fix now an arbitrary $\Lambda < \infty$ and apply E_Λ to get $u_\Lambda(t) = \int_0^t h^{(1)}(t-s)A_\Lambda u_\Lambda(s)ds$, where $A_\Lambda := E_\Lambda A = E_\Lambda A E_\Lambda$, and we have used the formula $E_\Lambda A = A E_\Lambda$, $E_\Lambda^2 = E_\Lambda$. Since A_Λ is a bounded linear operator, it follows that $u_\Lambda = 0$. Since Λ is arbitrary, this implies $u = 0$.

Existence of the solution requires special assumption on f , u_0 and u_1 . Since usually f , u_0 , u_1 are at our disposal when we study the inverse problem, assumption $H1$ is not restrictive and is quite natural: if A is known, then E_Λ and H_Λ are known, and one can choose the data f , u_0 , u_1 in H_Λ . Moreover if the data are noisy, that is f , u_0 , u_1 are known up to a (known) error δ , i.e

$$\|u_{0,\delta} - u_0\| \leq \delta, \quad \|u_{1,\delta} - u_1\| \leq \delta, \quad \|f_\delta - f\|_{C([0,T];H)} \leq \delta, \quad (3.1)$$

then one can use the data $E_\Lambda f_\delta$, $E_\Lambda u_{0,\delta}$, $E_\Lambda u_{1,\delta}$ which satisfy $H1$. Since E_Λ is known, computation of $E_\Lambda f_\delta$, $E_\Lambda u_{0,\delta}$, $E_\Lambda u_{1,\delta}$ presents no difficulties. In these arguments we assume that A is given exactly.

If one wants to weaken assumption $H1$, one can allow the data to have a non-zero component in $H \ominus H_\Lambda$, but this component must have coefficients exponentially decaying as $\lambda \rightarrow +\infty$.

Let us summarize the results of this section:

Theorem 3.1. *Let $A = A^*$ be a possibly unbounded operator and let $h \in C([0, T])$. Then problem (1.4), (1.5) has at most one solution in $C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A))$. If in addition the hypothesis $H1$ holds, then problem (1.4), (1.5) has a solution in $C^2([0, T]; H_\Lambda)$ and this solution is unique.*

Recall now that, if $f \in C^1([0, T]; H)$, and $l \in C^1([0, T])$ with $l(0) = 0$, then the direct problem (1.12), (1.13) with $l(0) = 0$, is equivalent to the second-order Cauchy problem

$$u''(t) = \int_0^t h(t-s)Au(s)ds + f'(t), \quad 0 \leq t \leq T, \quad (3.2)$$

$$u(0) = u_0, \quad u'(0) = f(0), \quad (3.3)$$

where $h(t) = l'(t)$. Further assume

$$H2 \quad u_0 \in H_\Lambda, \quad f \in C^1([0, T]; H_\Lambda).$$

Then from (3.2), (3.3) and Theorem 3.1 we get the following theorem:

Theorem 3.2. *Let $A = A^*$ be a possibly unbounded operator and let $l \in C^1([0, T])$, $l(0) = 0$. Then problem (1.12), (1.13) has at most one solution in $C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A))$. If in addition the hypothesis $H2$ holds, then problem (1.12), (1.13) has a solution in $C^2([0, T]; H_\Lambda)$ and this solution is unique.*

4 A mixed problem for equation (1.4)

In this section the solution to (1.4) which satisfies the boundary conditions

$$u(0) = u_0, \quad u(T) = u_2, \quad (4.1)$$

is studied. Note that $u''(0) = f(0)$, as follows from (1.4) if $u \in C^2([0, T]; H)$.

We will show that, under suitable assumptions, the data

$$(A, \varphi, u_0, u_2, f, g) \quad (4.2)$$

determine the pair (u, h) uniquely.

Let us assume that the operator A does not depend on time, $A = A^*$ and $\{\varphi_j\}_{j=1}^{+\infty}$ is an orthonormal basis of H such that $A\varphi_j = \lambda_j\varphi_j$, $j \in \mathbf{N}$, $\{\lambda_j\}_{j=1}^{+\infty}$ being a positive nondecreasing sequence diverging to $+\infty$

If (1.4) is solvable in $C^2([0, T]; H)$, then

$$u(t) = \sum_{j=1}^{+\infty} \hat{u}_j(t) \varphi_j, \quad \hat{u}_j(t) = (u(t), \varphi_j), \quad 0 \leq t \leq T. \quad (4.3)$$

Hence the Fourier coefficients \hat{u}_j solve the *scalar* boundary value problems

$$\hat{u}_j''(t) - \lambda_j h * \hat{u}_j(t) = \hat{f}_j(t), \quad (4.4)$$

$$\hat{u}_j(0) = \hat{u}_{0,j} := (u_0, \varphi_j), \quad \hat{u}_j(T) = \hat{u}_{2,j} := (u_2, \varphi_j), \quad (4.5)$$

where $f * g(t) = \int_0^t f(t-s)g(s)ds$.

Let

$$\hat{u}_j'(0) = c_j, \quad \tilde{f}_j(t) = \int_0^t (t-s) \hat{f}_j(s) ds. \quad (4.6)$$

Then (4.4) implies

$$\hat{u}_j(t) = \hat{u}_{0,j} + tc_j + \tilde{f}_j(t) + \lambda_j h^{(1)} * \hat{u}_j(t), \quad 0 \leq t \leq T, \quad (4.7)$$

where

$$h^{(1)}(t) = \int_0^t (t-s)h(s) ds, \quad 0 \leq t \leq T. \quad (4.8)$$

Define the Volterra operators K_j , $j \in \mathbf{N}$, by the formulas

$$(I - \lambda_j H)^{-1} = I + \lambda_j K_j, \quad Hf := h^{(1)} * f, \quad K_j f := k_j * f, \quad \forall j \in \mathbf{N}. \quad (4.9)$$

Note that $h \in C([0, T])$ and $f \in C([0, T]; H)$ imply $k_j \in C([0, T])$ for all $j \in \mathbf{R}$.

Then (4.7) and (4.9) imply, for any $t \in [0, T]$,

$$\hat{u}_j(t) = \hat{u}_{0,j} + tc_j + \tilde{f}_j(t) + \lambda_j (k_j * 1)(t) \hat{u}_{0,j} + \lambda_j (k_j * t)(t) c_j + \lambda_j k_j * \tilde{f}_j(t). \quad (4.10)$$

To satisfy condition $\hat{u}_j(T) = \hat{u}_{2,j}$ for any $j \in \mathbf{N}$ it is sufficient to assume

$$\left| T + \lambda_j \int_0^T k_j(T-s)s \, ds \right| > 0, \quad \forall j \in \mathbf{N}. \quad (4.11)$$

Under such an assumption from (4.10) it easily follows

$$c_j = \frac{\hat{u}_{2,j} - \hat{u}_{0,j} - \tilde{f}_j(T) - \lambda_j \int_0^T k_j(T-s)[\hat{u}_{0,j} + \tilde{f}_j(s)] \, ds}{T + \lambda_j \int_0^T k_j(T-s)s \, ds}. \quad (4.12)$$

If

$$h(t) \geq 0, \quad 0 \leq t \leq T, \quad \lambda_j > 0, \quad \forall j \in \mathbf{N}, \quad (4.13)$$

then (4.9) shows that

$$K_j = \sum_{m=1}^{+\infty} \lambda_j^{m-1} H^m \iff k_j = \sum_{m=1}^{+\infty} \lambda_j^{m-1} (h^{(1)*})^{m-1} h^{(1)}, \quad (4.14)$$

so $k_j(t) \geq 0$ for all $t \in [0, T]$ and all $j \in \mathbf{N}$. Therefore condition (4.11) is satisfied. Therefore we have proved

Lemma 4.1. *If (4.13) holds, then, for any $j \in \mathbf{N}$, problem (4.4), (4.5) is solvable for any triplet $(\hat{u}_{0,j}, \hat{u}_{2,j}) \in \mathbf{R}^2$, $\hat{f}_j \in C([0, T])$, and its solution is unique.*

Remark 4.1. According to equation (4.3) condition (4.11) is satisfied if we assume, e.g., that the data fulfill the following inequalities

$$g(0)g'(t) < 0, \quad \lambda_0 g(0)[g'''(t) - \lambda_0 g'(t) - (f'(t), \varphi)] > 0, \quad 0 \leq t \leq T. \quad (4.15)$$

Indeed, it suffices to show that (4.15) implies $k_j(t) \geq 0$ for any $t \in [0, T]$. The first of the conditions in (4.15) and the definition of function p in (2.2) imply that the solution h to equation (2.3), rewritten as a fixed-point equation, is non-negative in $[0, T]$. Hence, the same holds for $h^{(1)}$ according to (4.8). As a consequence, since $\lambda_j > 0$ for all $j \in \mathbf{N}$, from (4.14) we immediately deduce our assertion.

In general, the solution to (4.4) may not exist if the denominator of (4.12) vanishes for some integer j_0 . For the solution to (4.4) to exist in this case it is necessary and sufficient that the numerator of (4.12) vanishes also. If this is the case, the solution exists but is not unique: it is of the form (4.10) with c_{j_0} arbitrary.

For the series (4.3), representing the solution to (1.4) and (4.1), to converge in $C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A))$ it is necessary and sufficient that

$$\sup_{0 \leq t \leq T} \sum_{j=1}^{+\infty} \{|\hat{u}_j''(t)|^2 + \lambda_j^2 |\hat{u}_j(t)|^2\} < +\infty, \quad (4.16)$$

where \hat{u}_j is defined by (4.10) and (4.12).

Condition (4.16) is equivalent to requiring

$$\sup_{0 \leq t \leq T} \sum_{j=1}^{+\infty} \lambda_j^2 |\hat{u}_j(t)|^2 < +\infty, \quad (4.17)$$

as follows from (4.4).

Condition (4.17) is trivially satisfied if, for example,

$$\hat{u}_j(T) = \hat{u}_{0,j} = \tilde{f}_j(s) = 0, \quad \text{for } j \geq J, \ 0 \leq s \leq T, \quad (4.18)$$

where J is an arbitrarily large, fixed integer.

Condition (4.18) is sufficient but not necessary for (4.17) to hold.

Let us derive a less restrictive sufficient condition for (4.17) to hold, which is close to a necessary one. Denote

$$|\hat{u}_{0,j}|^2 + T^2 |c_j|^2 + \sup_{0 \leq t \leq T} |\tilde{f}_j(t)|^2 := L_j, \quad \forall j \in \mathbf{N}. \quad (4.19)$$

Formula (4.10) implies

$$|\hat{u}_j(t)|^2 \leq 6 \left(1 + T \lambda_j^2 \int_0^T |k_j(t)|^2 dt \right) L_j, \quad \forall j \in \mathbf{N}, \quad (4.20)$$

where the Cauchy inequality and the following elementary inequalities were used:

$$\left(\sum_{j=1}^n a_j \right)^2 \leq n \left(\sum_{j=1}^n a_j^2 \right), \quad a_j \geq 0,$$

with $n = 2$ and $n = 3$.

Let us now estimate k_j . If we denote

$$T \int_0^T |h(t)| dt := M, \quad (4.21)$$

then

$$\sup_{0 \leq t \leq T} h^{(1)}(t) \leq M. \quad (4.22)$$

If we set

$$h_m(t) = \int_0^t h^{(1)}(t-s) h_{m-1}(s) ds, \quad h_1(t) = h^{(1)}(t), \quad 0 \leq t \leq T, \quad (4.23)$$

where $h^{(1)}(t)$ is defined in (4.8), then, by induction, one gets

$$h_m(t) \leq M \frac{(Mt)^{m-1}}{(m-1)!}, \quad 0 \leq t \leq T, \quad \forall m \in \mathbf{N}. \quad (4.24)$$

Therefore (4.14) implies

$$0 \leq k_j(t) \leq M \exp(\lambda_j Mt), \quad 0 \leq t \leq T, \quad \forall j \in \mathbf{N}, \quad (4.25)$$

and

$$\int_0^T |k_j(t)|^2 dt \leq M^2 \frac{\exp(2\lambda_j MT) - 1}{2\lambda_j M} \leq M \frac{\exp(2\lambda_j MT)}{2\lambda_j}, \quad \forall j \in \mathbf{N}. \quad (4.26)$$

Since the positive nondecreasing sequence $\{\lambda_j\}_{j=1}^{+\infty}$ diverges to $+\infty$, formulas (4.20) and (4.26) imply

$$\sup_{0 \leq t \leq T} |\hat{u}_j(t)|^2 \leq 6L_j[1 + 0.5MT\lambda_j \exp(2\lambda_j MT)]. \quad (4.27)$$

From (4.27) it follows that (4.17) holds if

$$\sum_{j=1}^{+\infty} L_j \lambda_j^3 \exp(2\lambda_j MT) < +\infty. \quad (4.28)$$

Condition (4.28) means, roughly speaking, that for the series (4.28) to converge the sequences $\{\hat{u}_{0,j}\}_{j=1}^{+\infty}$, $\{\hat{u}_{2,j}\}_{j=1}^{+\infty}$ and $\{\tilde{f}_j\}_{j=1}^{+\infty}$ related to the Fourier coefficients of the data must decay faster than $\{\lambda_j^{-3/2} \exp(-\lambda_j MT)\}$ as $j \rightarrow \infty$.

Condition (4.28) is not far from a necessary condition since the above estimates were not too crude.

Let us summarize the result:

Theorem 4.1. *If (4.13) and (4.28) hold, then, for any non-negative $h \in C([0, T])$ the solution of the problem (1.4) and (4.1) in $C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A))$ does exist and is unique.*

To conclude this section we consider the identification problem consisting of recovering the pair $u : [0, T] \rightarrow H$ and $l : [0, T] \rightarrow \mathbf{R}$ satisfying the first-order Cauchy problem

$$u'(t) = l * Au(t) + f(t), \quad 0 \leq t \leq T, \quad (4.29)$$

$$u(T) = u_2. \quad (4.30)$$

as well as the extra data

$$g(t) = (u(t), \varphi), \quad l(0) = 0. \quad (4.31)$$

The assumptions about A are the same as at the beginning of this section, and $f \in C^1([0, T]; H)$, $g \in C^3([0, T])$.

Moreover, recalling (2.12), we assume that the data satisfy the conditions

$$g'(0) = (f(0), \varphi), \quad g''(0) - (f'(0), \varphi) = 0, \quad g(T) = (u_2, \varphi), \quad g(0) \neq 0. \quad (4.32)$$

Recall now that, since $l(0) = 0$, problem (4.29), (4.30) is equivalent to the second-order Cauchy problem

$$u''(t) = \int_0^t h(t-s)Au(s) ds + f'(t), \quad 0 \leq t \leq T, \quad (4.33)$$

$$u'(0) = f(0), \quad u(T) = u_2, \quad (4.34)$$

where $h(t) = l'(t)$.

Observe that the present problem differs from the one just studied only by the initial condition: $u'(0) = f(0)$ replaces $u(0) = u_0$. Moreover, since we have no initial condition $u(0) = u_0$, we need the explicit requirement $g(0) \neq 0$ (cf. (4.32)) to ensure that equation (2.10) is actually of the second kind.

Reasoning as at the beginning of this section, we easily get that formula (4.7) is now replaced by

$$\begin{aligned}\widehat{u}_j(t) &= c_j + t\widehat{f}_j(0) + \int_0^t (t-s)\widehat{f}_j'(s) ds + \lambda_j h^{(1)} * \widehat{u}_j(t) \\ &= c_j + \int_0^t \widehat{f}_j(s) ds + \lambda_j h^{(1)} * \widehat{u}_j(t),\end{aligned}\tag{4.35}$$

where c_j stands for the unknown value $\widehat{u}_j(0)$ and, in our case,

$$h^{(1)}(t) = \int_0^t (t-s)l'(s) ds = \int_0^t l(s) ds, \quad 0 \leq t \leq T.\tag{4.36}$$

Introducing the kernels k_j defined in (4.9) finally we get the representation formulas

$$\widehat{u}_j(t) = c_j + 1 * \widehat{f}_j(t) + c_j \lambda_j (k_j * 1)(t) + \lambda_j k_j * (1 * \widehat{f}_j(t)).\tag{4.37}$$

Moreover, the solvability condition (4.11) changes to

$$\left| 1 + \lambda_j \int_0^T k_j(s) ds \right| > 0, \quad \forall j \in \mathbf{N}.\tag{4.38}$$

Using (4.38), we obtain

$$c_j = \frac{\widehat{u}_{2,j} - \int_0^T \widetilde{f}_j(s) \left[1 + \lambda_j \int_0^{T-s} k_j(\sigma) d\sigma \right] ds}{1 + \lambda_j \int_0^T k_j(s) ds}, \quad \forall j \in \mathbf{N}.\tag{4.39}$$

Remark 4.2. According to equation (2.7) condition (4.38) is satisfied if we assume, e.g., that the data fulfill the following inequalities

$$g(0)g'(t) < 0, \quad \lambda_0 g(0)[g''(t) - (f'(t), \varphi)] > 0, \quad 0 \leq t \leq T.\tag{4.40}$$

Indeed, (4.40) implies that the solution l to equation (2.10) is non-negative in $[0, T]$. Then (4.36) implies $h^{(1)}(t) \geq 0$ for all $t \in [0, T]$. Finally, from (4.14) we deduce that k_j is non-negative in $[0, T]$ for any $j \in \mathbf{N}$.

To summarize our basic result for first-order integro-differential equations we need the notation

$$|\widehat{u}_{2,j}|^2 + |c_j|^2 + \sup_{0 \leq t \leq T} |\widetilde{f}_j(t)|^2 := L_j, \quad \forall j \in \mathbf{N}.\tag{4.41}$$

Theorem 4.2. *If $f \in C^1([0, T])$ and (4.28), (4.32) hold, then, for any non-negative $l \in C^1([0, T])$ the solution u of the problem (4.29), (4.30) in $C^2([0, T]; H) \cap C([0, T]; \mathcal{D}(A))$ does exist and is unique.*

5 Applications

Example 1. We apply the previous abstract result for second-order integro-differential equations to the identification problem: *determine two functions $u : [0, T] \times \Omega \rightarrow \mathbf{R}$ and $h : [0, T] \rightarrow \mathbf{R}$ satisfying the following equations for some $m \in \{0, 1\}$:*

$$\begin{aligned} D_t^2 u(t, x) &= \int_0^t h(t-s) \mathcal{A}(x, D_x) u(s, x) ds \\ &\quad + f(t, x), \quad (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (5.1)$$

$$u(mT, x) = u_{2m}(x), \quad D_t u(0, x) = u_1(x), \quad x \in \Omega, \quad (5.2)$$

$$\mathcal{B}u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (5.3)$$

$$\int_{\Omega} \varphi(x) u(t, x) dx = g(t), \quad 0 \leq t \leq T. \quad (5.4)$$

Here Ω denotes a bounded domain in \mathbf{R}^n with boundary $\partial\Omega$ of class $C^{1,1}$, while

$$\mathcal{A}(x, D_x)u = - \sum_{i,j=1}^n D_{x_i}(a_{i,j}(x) D_{x_j} u) + a_{0,0}(x)u, \quad (5.5)$$

$$\mathcal{A}(x, D_x)\varphi(x) = \lambda_0 \varphi(x), \quad \lambda_0 \neq 0. \quad (5.6)$$

Assume that:

$$\overline{a_{i,j}(x)} = a_{j,i}(x) = a_{i,j}(x) \in C^{0,1}(\overline{\Omega}), \quad i, j = 1, \dots, n, \quad a_{0,0} \in L^\infty(\Omega), \quad (5.7)$$

$$\sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \geq \mu \sum_{j=1}^n |\xi_j|^2 > 0, \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbf{R}^n. \quad (5.8)$$

Choose, for example,

$$\mathcal{B}u(x) = u(x), \quad x \in \partial\Omega, \quad (5.9)$$

and

$$\mathcal{D}(A) = \{u \in H^2(\Omega) : u = 0 \text{ on } \partial\Omega\}, \quad Au(x) = \mathcal{A}(x, D_x)u(x), \quad u \in \mathcal{D}(A). \quad (5.10)$$

Remark 5.1. Equation (5.1) is not of standard type, elliptic or hyperbolic. Also when the kernel h has a fixed sign, the behavior of the solution can be very wild (cf. section 4). At any rate, the behavior strongly depends on the type of the prescribed conditions: initial, boundary or mixed ones.

The results of sections 1–4 are applicable to problems (5.1)–(5.4) with $m = 0$ or $m = 1$ and $H = L^2(\Omega)$. Such results ensure, under explicit conditions on the data, the existence

and the uniqueness of a solution (u, h) to (5.1)–(5.4) and give an algorithm for recovery of h from the data.

Note that the existence of the non-zero eigenvalue of A^* follows from the selfadjointness of A and the known results about the eigenvalues of elliptic operators.

Example 2. We apply the previous abstract result for first-order integro-differential equations to the following identification problem: *determine two functions $u : [0, T] \times \Omega \rightarrow \mathbf{R}$ and $l : [0, T] \rightarrow \mathbf{R}$ satisfying the following equations for some (fixed) $m \in \{0, 1\}$:*

$$D_t u(t, x) = \int_0^t l(t-s) \mathcal{A}(x, D_x) u(s, x) ds + f(t, x), \quad (t, x) \in [0, T] \times \Omega, \quad (5.11)$$

$$u(mT, x) = \hat{u}_{2m}(x), \quad x \in \Omega, \quad (5.12)$$

$$\mathcal{B}u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (5.13)$$

$$\int_{\Omega} \varphi(x) u(t, x) dx = g(t), \quad 0 \leq t \leq T, \quad l(0) = 0. \quad (5.14)$$

Here Ω and $\mathcal{A}(x, D_x)$ enjoy the same properties as in Example 1.

The results of sections 1–4 are applicable to problems (5.11)–(5.13) with $m = 0$ or $m = 1$ and $H = L^2(\Omega)$. Such results ensure, under explicit conditions on the data, the existence and the uniqueness of a solution (u, h) to (5.11)–(5.14) and give an algorithm for recovery of l from the data.

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